MATH 312
Lecture 2: Trinity

(Converts system of linear equations to EASY triangular system.)

COUNTING SOLUTIONS OF LINEAR SYSTEMS

Gaussian Elimination (start with $n=1$)

1. Rearrange equations so that the coefficient of n-th variable in the n-th equation is non-zero (this coefficient is called the PIVOT)

2. Use the pivot to clear out the n-th variable from all equations BELOW the n-th one

3. If not yet triangular, goto 1 with $n \leftarrow n+1$

iterate

Example System with 3 equations in 3 variables:

$$\boxed{\begin{aligned} x+y+z &= 4 \\ x-2y-2z &= 7 \\ 2x-3y+3z &= -5 \end{aligned}} \quad \begin{aligned} &\text{— E1} \\ &\text{— E2} \\ &\text{— E3} \end{aligned}$$

Apply algorithm:

1st iteration

Pivot $= \boxed{1} =$ coeff of x in E1.

So, $E2' = E2 - E1$
and $E3' = E3 - 2E1$

This gives

$$\begin{aligned} x+y+z &= 4 && E1 \\ -3y-3z &= 3 && E2' \\ -5y+z &= -13 && E3' \end{aligned}$$

look ma, no x

2nd iteration

Pivot $= -3 =$ coeff of y in $E2'$.
So, $E3'' = E3' - \frac{5}{3} E2'$

This gives:

$$\boxed{\begin{aligned} x + y + z &= 4 \\ -3y - 3z &= 3 \\ 6z &= -18 \end{aligned}}$$

E1
E2'
E3''

No y, either

Now, we Back-substitute:

$z = -3$, so $y = 2$, so $x = 5$
(from E3'') (from E2') (from E1)

Ans: $\boxed{(x, y, z) = (5, 2, -3)}$

Today we will try this algorithm on 3 other systems of linear equations:

SYS I

$$\boxed{\begin{aligned} x + y + z &= 4 \\ 2x + 2y - 2z &= 7 \\ 2x - y - z &= 11 \end{aligned}}$$

SYS II

$$\boxed{\begin{aligned} x + y + z &= 4 \\ x - 2y - 2z &= 7 \\ 2x - y - z &= 10 \end{aligned}}$$

SYS III

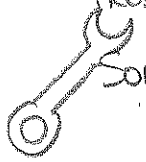
$$\boxed{\begin{aligned} x + y + z &= 4 \\ x - 2y - 2z &= 7 \\ 2x - y - z &= 11 \end{aligned}}$$

and see what happens ...

SYS I: Apply 1st iteration:

Pivot = 1, etc. We get:

$$\boxed{\begin{aligned} x+y+z&=4\\ -4z&=-1\\ -3y-3z&=3 \end{aligned}}$$



PROBLEM: Second equation has a zero y-coefficient!

SOLUTION: Rearrange! Swap second and third equations, continue as before. (Finish this at home)

SYS II: Apply 1st iteration, get

$$\boxed{\begin{aligned} x+y+z&=4 \quad & E1\\ -3y-3z&=3 \quad & E2'\\ -3y-3z&=2 \quad & E3' \end{aligned}}$$

Instead of continuing, note that E2′ and E3′ can not BOTH be true! This system has NO SOLUTION.

SYS III: Now, we get

$$\boxed{\begin{aligned} x+y+z&=4 \quad & E1\\ -3y-3z&=3 \quad & E2'\\ -3y-3z&=3 \quad & E3' \end{aligned}}$$

(one for each z)

And the second iteration WIPES OUT E3′ completely! This system has ∞-ly many solutions!

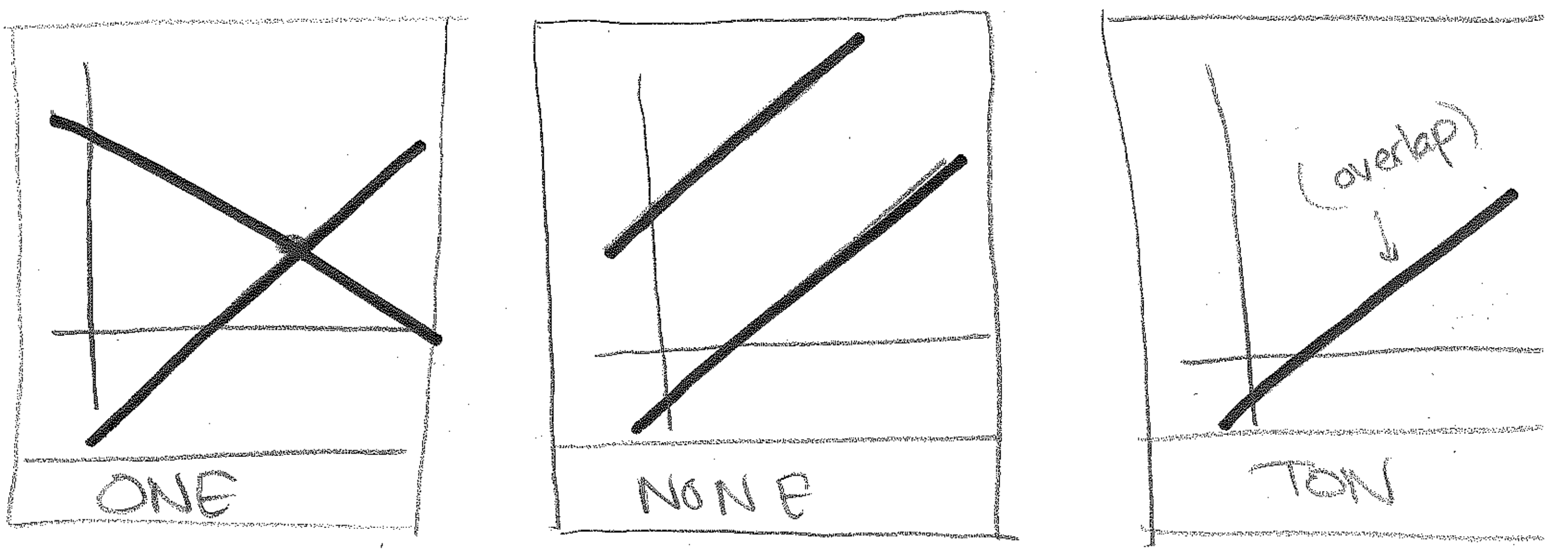
MAIN IDEA

These are the only 3 possibilities!! A given system of linear equations can only have $\{0, 1, \infty\}$ solutions!

none one f__ton

In the 2×2 case, we can see <u>WHY</u> using the <u>geometric interpretation</u> of solutions as intersection of lines!

There are ONLY THREE possibilities for intersections of "2" lines in the PLANE:



(most likely) (less likely) (least likely)

One intersection: generic case

No intersection: lines are parallel but disjoint

∞ Intersections: lines are THE SAME!